

Durham Research Online

Deposited in DRO:

12 December 2019

Version of attached file:

Accepted Version

Peer-review status of attached file:

Peer-reviewed

Citation for published item:

Boegli, Sabine (2017) 'Schrödinger operator with non-zero accumulation points of complex eigenvalues.', Communications in mathematical physics., 352 (2). pp. 629-639.

Further information on publisher's website:

<https://doi.org/10.1007/s00220-016-2806-5>

Publisher's copyright statement:

This is a post-peer-review, pre-copyedit version of an article published in Communications in mathematical physics. The final authenticated version is available online at: <https://doi.org/10.1007/s00220-016-2806-5>

Additional information:

Use policy

The full-text may be used and/or reproduced, and given to third parties in any format or medium, without prior permission or charge, for personal research or study, educational, or not-for-profit purposes provided that:

- a full bibliographic reference is made to the original source
- a [link](#) is made to the metadata record in DRO
- the full-text is not changed in any way

The full-text must not be sold in any format or medium without the formal permission of the copyright holders.

Please consult the [full DRO policy](#) for further details.

SCHRÖDINGER OPERATOR WITH NON-ZERO ACCUMULATION POINTS OF COMPLEX EIGENVALUES

SABINE BÖGLI

ABSTRACT. We study Schrödinger operators $H = -\Delta + V$ in $L^2(\Omega)$ where Ω is \mathbb{R}^d or the half-space \mathbb{R}_+^d , subject to (real) Robin boundary conditions in the latter case. For $p > d$ we construct a non-real potential $V \in L^p(\Omega) \cap L^\infty(\Omega)$ that decays at infinity so that H has infinitely many non-real eigenvalues accumulating at every point of the essential spectrum $\sigma_{\text{ess}}(H) = [0, \infty)$. This demonstrates that the Lieb-Thirring inequalities for selfadjoint Schrödinger operators are no longer true in the non-selfadjoint case.

1. INTRODUCTION

In three seminal papers [15, 16, 17] from the 1960s, Pavlov studied Schrödinger operators $H = -\Delta + V$ in $L^2(0, \infty)$ with real-valued rapidly decaying potentials V , subject to a non-selfadjoint Robin boundary condition $f'(0) = hf(0)$ for some $h \in \mathbb{C}$. In contrast to the selfadjoint case, for non-real h the discrete eigenvalues are complex and can, in principle, accumulate at a *non-zero* point of the essential spectrum $[0, \infty)$. Using inverse spectral theory, Pavlov proved the existence of a potential V and a boundary condition so that H has infinitely many non-real eigenvalues that accumulate at a prescribed point λ of the essential spectrum $\sigma_{\text{ess}}(H) = [0, \infty)$. He further studied the structure of the set of accumulation points. Since then, it has been an open question whether these results can be modified so that the non-selfadjointness is not coming from the boundary conditions but from a *non-real* potential V .

The aim of the present paper is to fill this gap by proving the following two results. In the first theorem we address non-selfadjoint Schrödinger operators in $L^2(\mathbb{R}^d)$ for any dimension $d \in \mathbb{N}$.

Theorem 1. *Let $p > d$ and $\mathcal{E} > 0$. There exists $V \in L^\infty(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ with $\max\{\|V\|_\infty, \|V\|_p\} \leq \mathcal{E}$ that decays at infinity so that the Schrödinger operator*

$$H := -\Delta + V, \quad \mathcal{D}(H) := W^{2,2}(\mathbb{R}^d),$$

has infinitely many eigenvalues in the open lower complex half-plane that accumulate at every point in $[0, \infty)$.

In the second main result we replace the whole Euclidean space \mathbb{R}^d by the half-space $\mathbb{R}_+^d := \{x = (x_1, \dots, x_d)^t \in \mathbb{R}^d : x_d > 0\}$ and impose (real) Robin boundary conditions.

Theorem 2. *Let $p > d$ and $\mathcal{E} > 0$, and let $\phi \in [0, \pi)$. There exists $V \in L^\infty(\mathbb{R}_+^d) \cap L^p(\mathbb{R}_+^d)$ with $\max\{\|V\|_\infty, \|V\|_p\} \leq \mathcal{E}$ that decays at infinity so that the Schrödinger operator*

$$H := -\Delta + V, \quad \mathcal{D}(H) := \{f \in W^{2,2}(\mathbb{R}_+^d) : \cos(\phi)\partial_{x_d}f + \sin(\phi)f = 0 \text{ on } \partial\mathbb{R}_+^d\},$$

has infinitely many eigenvalues in the open lower complex half-plane that accumulate at every point in $[0, \infty)$.

Date: October 26, 2016.

2010 Mathematics Subject Classification. 35J10, 81Q12.

Key words and phrases. Non-selfadjoint Schrödinger operator, complex potential, accumulation point of eigenvalues, Lieb-Thirring inequalities.

Theorem 1 is also relevant in the context of *Lieb-Thirring inequalities* (after Lieb and Thirring [14], see also [12] for an overview) and their (possible) generalisation to complex potentials [8, 13, 5]. In the selfadjoint case the Lieb-Thirring inequalities state that, if

$$p \geq \frac{d}{2} \quad \text{for } d \geq 3; \quad p > 1 \quad \text{for } d = 2; \quad p \geq 1 \quad \text{for } d = 1, \quad (1)$$

then there exists $C_{d,p} > 0$ so that for every real $V \in L^p(\mathbb{R}^d)$ the negative eigenvalues of the Schrödinger operator $H = -\Delta + V$ satisfy

$$\sum_{\lambda \in \sigma(H) \setminus [0, \infty)} |\lambda|^{p-\frac{d}{2}} \leq C_{d,p} \|V\|_p^p \quad (2)$$

where in the sum each eigenvalue is repeated according to its algebraic multiplicity. In fact, the inequality remains true if V on the right hand side is replaced by the negative part $V_- := \max\{0, -V\}$. Now Theorem 1 demonstrates that, if $p > d$, an inequality like (2) cannot hold in the non-selfadjoint case since, for the constructed V in Theorem 1, the left hand side is infinite whereas the right hand side is finite (and, in fact, arbitrarily small). The sharpness of $p > d$ (in relation to p in (1)) is discussed in Remark 1 below. For possible modifications of Lieb-Thirring inequalities see [6] and the references therein.

Theorem 1 is proved in Section 2, and Theorem 2 in Section 3. In contrast to Pavlov's inverse spectral theory approach using an elaborate analysis of Weyl m -functions, our proofs are constructive. For both $\Omega = \mathbb{R}^d$ and $\Omega = \mathbb{R}_+^d$ the proof relies on the following two main ingredients (see Lemmas 1, 2 and 3, 4 for the precise formulation):

- (I) For an arbitrary $\lambda \in (0, \infty)$ we construct $V_0 \in L^\infty(\Omega) \cap L^p(\Omega)$ with arbitrarily small $\|V_0\|_\infty$, $\|V_0\|_p$ and that decays at infinity so that $-\Delta + V_0$ in $L^2(\Omega)$ has an eigenvalue μ close to λ .
- (II) For two potentials $V_1 \in L^\infty(\Omega)$, $V_2 \in L^\infty(\mathbb{R}^d)$ decaying at infinity, consider the corresponding Schrödinger operators

$$H_1 := -\Delta + V_1 \quad \text{in } L^2(\Omega), \quad H_2 := -\Delta + V_2 \quad \text{in } L^2(\mathbb{R}^d),$$

and assume that there exists $\mu \in \sigma(H_2) \setminus \sigma(H_1)$. If we shift V_2 in direction of the d -th coordinate vector e_d to $V_2(\cdot - te_d)$ for a sufficiently large $t > 0$, then $H_1 + \chi_\Omega V_2(\cdot - te_d)$ in $L^2(\Omega)$ has an eigenvalue μ_t close to μ .

The potential V in Theorems 1, 2 is then an infinite sum of functions V_j , $j \in \mathbb{N}$, that we construct inductively using (I) and (II) above.

Since we do not know the exact value of the “sufficiently large” shift t in (II), we cannot control the exact decay rate of V at infinity. For $\Omega = \mathbb{R}^3$ or $\Omega = (0, \infty)$, subject to the boundary condition $f(0) = 0$ or $f'(0) = hf(0)$, $h \in \mathbb{C}$, in the half-line case, Pavlov [15] proved that if

$$\exists \varepsilon > 0 : \quad \sup_{x \in \Omega} |V(x)| e^{\varepsilon \sqrt{|x|}} < \infty, \quad (3)$$

then $-\Delta + V$ in $L^2(\Omega)$ has only finitely many eigenvalues. Therefore, the potential V in Theorem 1 (for $d = 3$) and Theorem 2 (for $d = 1$) has to decay so slow to violate (3). The condition (3) for $\Omega = (0, \infty)$ is sharp; Pavlov [16] proved that it cannot be relaxed to $\sup_{x \in (0, \infty)} |V(x)| e^{\varepsilon x^\beta} < \infty$ for any $\beta \in (0, \frac{1}{2})$. For an arbitrary odd dimension d , see [9] and the references therein for conditions guaranteeing a finite number of eigenvalues. In addition, in [18] are conditions, for an arbitrary $d \geq 2$, that prevent a dissipative Schrödinger operator (where $\text{Im } V \leq 0$) to have discrete eigenvalues accumulating at zero.

We employ the following notation and conventions. Let $\mathbb{N} := \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The open ball in \mathbb{R}^d with radius $r > 0$ around $v \in \mathbb{R}^d$ is $B(v, r) := \{x \in \mathbb{R}^d : |x - v| < r\}$, and analogously $B(z, r) \subset \mathbb{C}$ denotes the open disk of radius $r > 0$ around $z \in \mathbb{C}$. For a subset $\Lambda \subset \mathbb{C}$ the complex conjugated set is $\Lambda^* := \{\bar{\lambda} : \lambda \in \Lambda\}$, and for $z \in \mathbb{C}$ its distance to Λ is $\text{dist}(z, \Lambda) := \inf_{\lambda \in \Lambda} |z - \lambda|$. Take a domain $\Omega \subset \mathbb{R}^d$ and $p \in [1, \infty]$. A function $f \in L^p(\Omega)$ is viewed as an element of $L^p(\mathbb{R}^d)$ by extending it by zero outside Ω , with L^p norm $\|f\|_p$; conversely, if we multiply a function $g \in L^p(\mathbb{R}^d)$ with the characteristic function χ_Ω of Ω , then $\chi_\Omega g \in L^p(\Omega)$. If not specified by an index, the norm $\|\cdot\|$ always refers to the one of the Hilbert space $L^2(\mathbb{R}^d)$. The operator domain, spectrum and resolvent set of an operator H are denoted by $\mathcal{D}(H)$, $\sigma(H)$ and $\varrho(H)$, and the Hilbert space adjoint operator is H^* . An identity operator is denoted by I , and scalar multiples λI for $\lambda \in \mathbb{C}$ are written as λ . Analogously, in $L^2(\mathbb{R}^d)$ the operator of multiplication with an $L^\infty(\mathbb{R}^d)$ function V is simply V ; its adjoint operator is the multiplication operator with the complex conjugated function \bar{V} . Weak convergence in $L^2(\mathbb{R}^d)$ is denoted by $f_n \xrightarrow{w} f$, and strong operator convergence is $H_n \xrightarrow{s} H$.

2. SCHRÖDINGER OPERATOR IN $L^2(\mathbb{R}^d)$

Throughout this section, all operator domains are $W^{2,2}(\mathbb{R}^d)$. The functions V_j , $j \in \mathbb{N}$, mentioned in the introduction will be of the form

$$U_{c,t,a}(x) := \begin{cases} c, & x \in B(te_d, a), \\ -\frac{(d-3)(d-1)}{4|x-te_d|^2}, & x \in \mathbb{R}^d \setminus B(te_d, a), \end{cases}$$

where $c \in \mathbb{C}$, $t \in \mathbb{R}$ and $a > 0$. Note that in dimension $d = 1$ and $d = 3$ the function $U_{c,t,a}$ vanishes outside the ball $B(te_d, a)$.

Before we study finite or infinite sums, we reduce our attention to a potential of the form $U_{c,t,a}$.

Lemma 1. *Let $\lambda \in (0, \infty)$ and $p > d$. For any $\varepsilon, \delta, r > 0$ there exist $a > 0$, $c \in \mathbb{C}$ and $\mu \in \mathbb{C}$ with $\text{Im } \mu < 0$ such that, for every $t \in \mathbb{R}$,*

$$\|U_{c,t,a}\|_p < \varepsilon, \quad \|U_{c,t,a}\|_\infty < \delta, \quad |\mu - \lambda| < r,$$

and μ is an eigenvalue of $-\Delta + U_{c,t,a}$.

Proof. Define $\nu := \sqrt{\lambda} > 0$ and

$$a_m := \frac{\frac{d\pi}{4} + \pi m}{\nu} > 0, \quad m \in \mathbb{N}_0. \quad (4)$$

For $m \in \mathbb{N}_0$ let $\eta_m > 0$ be the unique solution of

$$\eta_m e^{2\eta_m a_m} = \nu. \quad (5)$$

Note that $a_m \rightarrow \infty$ and $\eta_m \rightarrow 0$ as $m \rightarrow \infty$. We set

$$\tau_m := \nu + i\eta_m, \quad m \in \mathbb{N}_0,$$

and

$$k_m := -i \frac{J_{\frac{d}{2}-2}(\tau_m a_m)}{J_{\frac{d}{2}-1}(\tau_m a_m)} \tau_m + \frac{i(d-3)}{2a_m}, \quad m \in \mathbb{N}_0, \quad (6)$$

where J_n is the Bessel function of the first kind of order n (see [2, Chapter 9]). It satisfies

$$J'_n(z) = J_{n-1}(z) - \frac{nJ_n(z)}{z}, \quad z^2 J''_n(z) + zJ'_n(z) = (n^2 - z^2)J_n(z),$$

see [2, Equation 9.1.27]. For a fixed $m \in \mathbb{N}_0$, define the function

$$g_m(r) := \begin{cases} \frac{e^{ik_m a_m}}{\sqrt{a_m} J_{\frac{d}{2}-1}(\tau_m a_m)} \frac{\tau_m^{\frac{d}{2}-1}}{2^{\frac{d}{2}-1} \Gamma(\frac{d}{2})}, & r = 0, \\ \frac{e^{ik_m a_m}}{\sqrt{a_m} J_{\frac{d}{2}-1}(\tau_m a_m)} \frac{J_{\frac{d}{2}-1}(\tau_m r)}{r^{\frac{d}{2}-1}}, & 0 < r \leq a_m, \\ \frac{e^{ik_m r}}{r^{\frac{d-1}{2}}}, & r > a_m. \end{cases}$$

Using (6) and [2, Equation 9.1.10], one may check that both g_m and g'_m are continuous; for small $r > 0$ we expand $g_m(r) = g_m(0) + \mathcal{O}(r^2)$, hence $\lim_{r \rightarrow 0} g'_m(r) = 0$. Let $t \in \mathbb{R}$ be arbitrary. Then $f_m(x) := g_m(|x - te_d|)$, $x \in \mathbb{R}^d$, belongs to $W_{\text{loc}}^{2,2}(\mathbb{R}^d)$ and

$$\begin{aligned} -\Delta f_m(x) &= -g_m''(|x - te_d|) - \frac{d-1}{|x - te_d|} g'_m(|x - te_d|) \\ &= \begin{cases} \tau_m^2 f_m(x), & 0 < |x - te_d| \leq a_m, \\ k_m^2 f_m(x) + \frac{(d-3)(d-1)}{4|x - te_d|^2} f_m(x), & |x - te_d| > a_m. \end{cases} \end{aligned}$$

Hence

$$-\Delta f_m + U_{c_m, t, a_m} f_m = \mu_m f_m \quad \text{with} \quad \mu_m := k_m^2, \quad c_m := k_m^2 - \tau_m^2.$$

In order to ensure $f_m \in W^{2,2}(\mathbb{R}^d) = \mathcal{D}(-\Delta + U_{c_m, t, a_m})$, we need $\text{Im } k_m > 0$. We use the asymptotics of the Bessel function for $z \in \mathbb{C}$ with $|\arg z| < \pi$ and large $|z|$ (see [2, Equation 9.2.1]),

$$J_n(z) = \sqrt{\frac{2}{\pi z}} \left(\cos \left(z - \frac{(2n+1)\pi}{4} \right) + e^{|\text{Im } z|} \mathcal{O}(|z|^{-1}) \right).$$

A straight forward calculation reveals that, if

$$\text{Re } z \in \frac{(n+1)\pi}{2} + \pi\mathbb{Z}, \quad \text{Im } z > 0, \quad (7)$$

then for large $|z|$ we have

$$\begin{aligned} \frac{J_{n-1}(z)}{J_n(z)} &= -\frac{e^{-\text{Im } z} + i e^{\text{Im } z} + e^{\text{Im } z} \mathcal{O}(|z|^{-1})}{i e^{-\text{Im } z} + e^{\text{Im } z} + e^{\text{Im } z} \mathcal{O}(|z|^{-1})} \\ &= -2e^{-2\text{Im } z} + i(e^{-4\text{Im } z} - 1) + \mathcal{O}(|z|^{-1}). \end{aligned}$$

The point $z = \tau_m a_m$ satisfies (7) for $n = \frac{d}{2} - 1$, and hence, for large m , (6) yields

$$\begin{aligned} k_m &= -i\tau_m \left(-2e^{-2\text{Im } \tau_m a_m} + i(e^{-4\text{Im } \tau_m a_m} - 1) + \mathcal{O}(|\tau_m a_m|^{-1}) \right) + \mathcal{O}(a_m^{-1}) \\ &= -\nu(1 - e^{-4\eta_m a_m}) - 2\eta_m e^{-2\eta_m a_m} \\ &\quad + i(2\nu e^{-2\eta_m a_m} - \eta_m(1 - e^{-4\eta_m a_m})) + \mathcal{O}(a_m^{-1}). \end{aligned}$$

Using that (5) implies $e^{-2\eta_m a_m} = \frac{\eta_m}{\nu}$ and $a_m = \frac{\ln(\nu/\eta_m)}{2\eta_m}$, we arrive at

$$k_m = -\nu + i\eta_m \left(1 + \mathcal{O} \left(\left(\ln \frac{\nu}{\eta_m} \right)^{-1} \right) \right).$$

Since $\eta_m > 0$ and $\ln(\nu/\eta_m)^{-1} \rightarrow 0$ as $m \rightarrow \infty$, we conclude that $\text{Im } k_m > 0$ for all sufficiently large $m \in \mathbb{N}_0$. In addition, for large $m \in \mathbb{N}_0$ the eigenvalue $\mu_m = k_m^2$ satisfies

$$\mu_m = \lambda - i2\nu\eta_m \left(1 + \mathcal{O} \left(\left(\ln \frac{\nu}{\eta_m} \right)^{-1} \right) \right),$$

and hence $\text{Im } \mu_m < 0$ for all sufficiently large $m \in \mathbb{N}_0$. One may check that

$$\mu_m - \lambda = \mathcal{O}(\eta_m), \quad c_m = k_m^2 - \tau_m^2 = \mathcal{O}(\eta_m)$$

converge to 0 as $m \rightarrow \infty$. Further note that

$$\begin{aligned} \|U_{c_m, t, a_m}\|_p^p &= \text{Vol}(B(0, 1)) \left(\frac{|c_m|^p a_m^d}{d} + \frac{|d-3|^p |d-1|^p}{4^p (2p-d) a_m^{2p-d}} \right) = \mathcal{O}\left(\eta_m^{p-d} \left(\ln \frac{\nu}{\eta_m}\right)^d\right), \\ \|U_{c_m, t, a_m}\|_\infty &= \max \left\{ |c_m|, \frac{|d-3| |d-1|}{4 a_m^2} \right\} = \mathcal{O}(\eta_m). \end{aligned}$$

Since $p > d$ by the assumptions, both norms converge to 0 as $m \rightarrow \infty$. Altogether, we see that the claim is satisfied if we set $a := a_m$, $c := c_m$, $\mu := \mu_m$ for a sufficiently large $m \in \mathbb{N}_0$. \square

Remark 1. In dimension $d = 1$ the assumption $p > d = 1$ of Lemma 1 is sharp. In fact, due to Abramov et al. [1], for every $V \in L^1(\mathbb{R})$ every eigenvalue $\mu \in \sigma(-d^2/dx^2 + V) \setminus [0, \infty)$ satisfies

$$|\mu|^{\frac{1}{2}} \leq \frac{1}{2} \|V\|_1; \quad (8)$$

hence $\varepsilon > 0$ cannot be chosen arbitrarily small as in Lemma 1. In addition, in Theorem 1 for $d = 1$ it is impossible to construct $V \in L^1(\mathbb{R}^d)$ since then (8) forces the non-real eigenvalues to lie in the disk $B(0, \mathcal{E}^2/4)$, so they cannot accumulate at every point in $[0, \infty)$.

For dimension $d \geq 2$ the sharpness of the assumption $p > d$ is directly related to the following conjecture of Laptev and Safronov [13]: For $p \in (\frac{d}{2}, d]$ there exists $C_{d,p} > 0$ such that

$$|\mu|^{p-\frac{d}{2}} \leq C_{d,p} \|V\|_p^p \quad (9)$$

for every $V \in L^p(\mathbb{R}^d)$ and every $\mu \in \sigma(-\Delta + V) \setminus [0, \infty)$. In [10] the conjecture was proved for *radial* potentials. Note that the potential in Lemma 1 is radial, so $p > d$ is sharp. In general (for non-radial potentials) the conjecture has been confirmed for $p \in (\frac{d}{2}, \frac{d+1}{2}]$ (see [7]) and is still open for $p \in (\frac{d+1}{2}, d]$. If the conjecture is false, then it may also be possible to modify Lemma 1 for a non-radial potential and hence prove Theorems 1, 2 for a $p \leq d$.

Lemma 2. Let $V_1, V_2 \in L^\infty(\mathbb{R}^d)$ be decaying at infinity and such that there exists $\mu \in \sigma(-\Delta + V_2) \setminus \sigma(-\Delta + V_1)$. Then there are

$$\mu_t \in \sigma(-\Delta + V_1 + V_2(\cdot - te_d)), \quad t > 0,$$

with $\mu_t \rightarrow \mu$ as $t \rightarrow \infty$.

Proof. First note that

$$\sigma(-\Delta + V_1 + V_2(\cdot - te_d)) = \sigma(-\Delta + V_1(\cdot + te_d) + V_2), \quad t > 0. \quad (10)$$

Next we prove that, for every $z \in \mathbb{C}$ with $\text{dist}(z, [0, \infty)) > \|V_1\|_\infty + \|V_2\|_\infty$, we have strong resolvent convergence

$$(-\Delta + V_1(\cdot + te_d) + V_2 - z)^{-1} \xrightarrow{s} (-\Delta + V_2 - z)^{-1}, \quad t \rightarrow \infty, \quad (11)$$

and the same holds for the adjoint operators. To this end, first note that a Neumann series argument yields

$$\begin{aligned}
z &\in \bigcap_{t>0} \varrho(-\Delta + V_1(\cdot + te_d) + V_2) \cap \varrho(-\Delta + V_2), \\
&\sup_{t>0} \|(-\Delta + V_1(\cdot + te_d) + V_2 - z)^{-1}\| \\
&\leq \|(-\Delta - z)^{-1}\| \sup_{t>0} \|(I + (V_1(\cdot + te_d) + V_2)(-\Delta - z)^{-1})^{-1}\| \\
&\leq \frac{1}{\text{dist}(z, [0, \infty))} \frac{1}{1 - \frac{\|V_1\|_\infty + \|V_2\|_\infty}{\text{dist}(z, [0, \infty))}} = \frac{1}{\text{dist}(z, [0, \infty)) - (\|V_1\|_\infty + \|V_2\|_\infty)}.
\end{aligned}$$

The space $C_0^\infty(\mathbb{R}^d)$ is dense in $W^{2,2}(\mathbb{R}^d)$ and hence a core of $-\Delta + V_2$. Let $f \in C_0^\infty(\mathbb{R}^d)$. Then $f \in W^{2,2}(\mathbb{R}^d)$, and the assumption $V_1(x) \rightarrow 0$ as $|x| \rightarrow \infty$ yields

$$\|(-\Delta + V_1(\cdot + te_d) + V_2)f - (-\Delta + V_2)f\| \leq \sup_{x \in (\text{supp} f + te_d)} |V_1(x)| \|f\| \rightarrow 0, \quad t \rightarrow \infty.$$

Now the strong resolvent convergence in (11) follows from [3, Theorem 3.1, Proposition 2.16 i)], and the strong resolvent convergence of the adjoint operators

$$(-\Delta + V_1(\cdot + te_d) + V_2)^* = -\Delta + \overline{V_1(\cdot + te_d)} + \overline{V_2}, \quad t > 0,$$

to $(-\Delta + V_2)^* = -\Delta + \overline{V_2}$ is proved analogously.

By [4, Theorem 2.3 i)], in the limit $t \rightarrow \infty$ the isolated eigenvalue $\mu \in \sigma(-\Delta + V_2) \setminus \sigma(-\Delta + V_1)$ is approximated by points $\mu_t \in \sigma(-\Delta + V_1(\cdot + te_d) + V_2)$, $t > 0$, provided that the so-called *limiting essential spectrum* satisfies

$$\mu \notin \sigma_{\text{ess}}((-\Delta + V_1(\cdot + te_d) + V_2)_{t>0}) \cup \sigma_{\text{ess}}(((-\Delta + V_1(\cdot + te_d) + V_2)^*)_{t>0})^*. \quad (12)$$

This, together with (10), then proves the claim. So it is left to prove (12).

By definition (see [4]), the point μ belongs to set on the right hand side of (12) only if there exist an infinite subset $I \subset (0, \infty)$ and $f_t \in W^{2,2}(\mathbb{R}^d)$, $t \in I$, with $\|f_t\| = 1$, $f_t \xrightarrow{w} 0$ and, in the limit $t \rightarrow \infty$,

$$\begin{aligned}
&\|(-\Delta + V_1(\cdot + te_d) + V_2 - \mu)f_t\| \rightarrow 0 \\
\text{or} \quad &\|(-\Delta + \overline{V_1(\cdot + te_d)} + \overline{V_2} - \bar{\mu})f_t\| \rightarrow 0.
\end{aligned} \quad (13)$$

It is easy to see that the latter implies that $\|f_t\|_{W^{1,2}(\mathbb{R}^d)}$, $t \in I$, are uniformly bounded. Since, for any $r > 0$, the space $W^{1,2}(B(0, r))$ is compactly embedded in $L^2(B(0, r))$ by the Rellich-Kondrachov theorem, the weak convergence $f_t \xrightarrow{w} 0$ implies $\|\chi_{B(0, r)} f_t\| \rightarrow 0$ and hence $\|\chi_{B(0, r)} V_2 f_t\| \rightarrow 0$ as $t \rightarrow \infty$. Moreover, the assumption $V_2(x) \rightarrow 0$ as $|x| \rightarrow \infty$ yields

$$\sup_{t>0} \|\chi_{\mathbb{R}^d \setminus B(0, r)} V_2 f_t\| \leq \sup_{|x|>r} |V_2(x)| \rightarrow 0, \quad r \rightarrow \infty.$$

Altogether, in the limit $t \rightarrow \infty$ we obtain $\|V_2 f_t\| \rightarrow 0$ and hence, by (13),

$$\begin{aligned}
&\|(-\Delta + V_1 - \mu)f_t(\cdot - te_d)\| = \|(-\Delta + V_1(\cdot + te_d) - \mu)f_t\| \rightarrow 0 \\
\text{or} \quad &\|(-\Delta + \overline{V_1} + \bar{\mu})f_t(\cdot - te_d)\| = \|(-\Delta + \overline{V_1(\cdot + te_d)} + \bar{\mu})f_t\| \rightarrow 0.
\end{aligned}$$

Therefore, in either case μ needs to belong to $\sigma(-\Delta + V_1) = \sigma(-\Delta + \overline{V_1})^*$, which is excluded by the assumptions. This proves the claim (12). \square

Now we are ready to prove the main result.

Proof of Theorem 1. Consider an enumeration of $(\mathbb{Q} \cap (0, \infty)) \times \mathbb{N}$, i.e. a bijective map

$$\mathbb{N} \ni n \mapsto (q_n, m_n)^t \in (\mathbb{Q} \cap (0, \infty)) \times \mathbb{N}.$$

Set $\gamma_0 := \infty$. By induction over $n \in \mathbb{N}$ we construct c_n, t_n, a_n and γ_n such that

$$H_n := -\Delta + \sum_{j=1}^n U_{c_j, t_j, a_j}$$

satisfies the following:

i) The norms of the functions are bounded by

$$\begin{aligned} \|U_{c_n, t_n, a_n}\|_p &< \varepsilon_n := \frac{6\mathcal{E}}{\pi^2 n^2}, \\ \|U_{c_n, t_n, a_n}\|_\infty &< \delta_n := \frac{6 \min\{\gamma_{n-1}, \mathcal{E}\}}{\pi^2 n^2}, \end{aligned} \quad (14)$$

and

$$\exists \mu_n \in \sigma(H_n) : \quad \text{Im } \mu_n < 0, \quad |\mu_n - q_n| < \frac{1}{2m_n}. \quad (15)$$

ii) We have $0 < \gamma_n \leq \gamma_{n-1}$ and for any $U_n \in L^\infty(\mathbb{R}^d)$ with $\|U_n\|_\infty < \gamma_n$ there is $\lambda_n \in \sigma(H_n + U_n)$ such that

$$|\lambda_n - \mu_n| < \text{dist}(\mu_n, [0, \infty)).$$

We start with $n = 1$. By Lemma 1 applied to

$$\lambda = q_1, \quad \varepsilon = \varepsilon_1, \quad \delta = \delta_1, \quad r = \frac{1}{2m_1}$$

and an arbitrary $t_1 \in \mathbb{R}$, there exist $c_1 \in \mathbb{C}$, $a_1 > 0$ and an eigenvalue satisfying (15) for $n = 1$. By [11, Theorems IV.2.14, 3.16], there exists γ_1 satisfying claim ii) for $n = 1$.

Now assume that for $j = 1, \dots, n-1$ the constants c_j, t_j, a_j and γ_j have been constructed. We construct c_n, t_n, a_n and γ_n so that H_n satisfies i) and ii). We apply Lemma 1 to

$$\lambda = q_n, \quad \varepsilon = \varepsilon_n, \quad \delta = \delta_n, \quad r = \min \left\{ \text{dist}(\lambda, \sigma(H_{n-1})), \frac{1}{4m_n} \right\}.$$

In this way we obtain $c_n \in \mathbb{C}$ and $a_n > 0$ such that, for any $t \in \mathbb{R}$, the Schrödinger operator $-\Delta + U_{c_n, t, a_n}$ has an eigenvalue $\mu \in \sigma(-\Delta + U_{c_n, t, a_n}) \setminus \sigma(H_{n-1})$ with $\text{Im } \mu < 0$ and

$$\|U_{c_n, t, a_n}\|_p < \varepsilon_n, \quad \|U_{c_n, t, a_n}\|_\infty < \delta_n, \quad |\mu - q_n| < \frac{1}{4m_n}.$$

Lemma 2 implies that, for $t_n := t$ sufficiently large, the operator $H_n = H_{n-1} + U_{c_n, t_n, a_n}$ has an eigenvalue μ_n with $\text{Im } \mu_n < 0$, $|\mu_n - \mu| < 1/(4m_n)$ and hence $|\mu_n - q_n| < 1/(2m_n)$. This proves claim i), and claim ii) follows again from [11, Theorems IV.2.14, 3.16].

Finally we prove that the potential

$$V := \sum_{j=1}^{\infty} U_{c_j, t_j, a_j}$$

satisfies the claims of the theorem. By Minkowski's inequality and (14),

$$\max\{\|V\|_p, \|V\|_\infty\} < \sum_{j=1}^{\infty} \max\{\varepsilon_j, \delta_j\} \leq \frac{6\mathcal{E}}{\pi^2} \sum_{j=1}^{\infty} \frac{1}{j^2} = \mathcal{E}.$$

Moreover, for $n \in \mathbb{N}$ the $L^\infty(\mathbb{R}^d)$ norm of $U_n := \sum_{j=n+1}^{\infty} U_{c_j, t_j, a_j}$ is estimated as

$$\|U_n\|_\infty < \sum_{j=n+1}^{\infty} \delta_j \leq \frac{6\gamma_n}{\pi^2} \sum_{j=n+1}^{\infty} \frac{1}{j^2} < \gamma_n.$$

So the above claim ii) implies for $H_n + U_n = H$ that

$$\exists \lambda_n \in \sigma(H) : \quad |\lambda_n - \mu_n| < \text{dist}(\mu_n, [0, \infty)).$$

Hence $\text{Im } \lambda_n < 0$ and

$$|\lambda_n - q_n| \leq |\lambda_n - \mu_n| + |\mu_n - q_n| < \text{dist}(\mu_n, [0, \infty)) + |\mu_n - q_n| < \frac{1}{m_n},$$

i.e. $\lambda_n \in B(q_n, \frac{1}{m_n})$, $n \in \mathbb{N}$. Now it is easy to see that every point in $[0, \infty)$, which is the closure of $\mathbb{Q} \cap (0, \infty)$, is an accumulation point of $\{\lambda_n : n \in \mathbb{N}\}$. \square

3. SCHRÖDINGER OPERATOR IN $L^2(\mathbb{R}_+^d)$

In this section we study Schrödinger operators on the half-space \mathbb{R}_+^d , and for the proof of Lemma 4 below also on the shifted half-space $\mathbb{R}_+^d + te_d$ for some $t \in \mathbb{R}$. We fix an angle $\phi \in [0, \pi)$ which determines the Robin boundary condition. Throughout this section, every operator in $L^2(\mathbb{R}_+^d + te_d)$ for some $t \in \mathbb{R}$ is assumed to have the operator domain

$$\{f \in W^{2,2}(\mathbb{R}_+^d + te_d) : \cos(\phi)\partial_{x_d}f + \sin(\phi)f = 0 \text{ on } \partial(\mathbb{R}_+^d + te_d)\},$$

and operators in $L^2(\mathbb{R})$ have domains $W^{2,2}(\mathbb{R})$.

The following result is almost the same as Lemma 1; note that here t is not arbitrary but needs to be sufficiently large, and the eigenvalue μ_t depends on t .

Lemma 3. *Let $\lambda \in (0, \infty)$ and $p > d$. For any $\varepsilon, \delta, r > 0$ there exist $a > 0$ and $c \in \mathbb{C}$ with*

$$\|U_{c,t,a}\|_p < \varepsilon, \quad \|U_{c,t,a}\|_\infty < \delta, \tag{16}$$

and such that, for every sufficiently large $t > 0$, the operator

$$-\Delta + \chi_{\mathbb{R}_+^d} U_{c,t,a} \quad \text{in} \quad L^2(\mathbb{R}_+^d)$$

has an eigenvalue μ_t with $\text{Im } \mu_t < 0$ and $|\mu_t - \lambda| < r$.

For the proof we use the following result, which is the analogue of Lemma 2.

Lemma 4. *Let $V_1 \in L^\infty(\mathbb{R}_+^d)$, $V_2 \in L^\infty(\mathbb{R}^d)$ be decaying at infinity, and define the operators*

$$H_1 := -\Delta + V_1 \quad \text{in} \quad L^2(\mathbb{R}_+^d), \quad H_2 := -\Delta + V_2 \quad \text{in} \quad L^2(\mathbb{R}^d).$$

Assume that there exists $\mu \in \sigma(H_2) \setminus \sigma(H_1)$. Then, for any $t > 0$, the operator

$$H_1 + \chi_{\mathbb{R}_+^d} V_2(\cdot - te_d) \quad \text{in} \quad L^2(\mathbb{R}_+^d)$$

has an eigenvalue μ_t with $\mu_t \rightarrow \mu$ as $t \rightarrow \infty$.

Proof. Define operators

$$H_{2,t} := -\Delta + \chi_{(\mathbb{R}_+^d - te_d)} V_2 \quad \text{in} \quad L^2(\mathbb{R}_+^d - te_d), \quad t > 0.$$

Note that

$$\sigma(H_1 + \chi_{\mathbb{R}_+^d} V_2(\cdot - te_d)) = \sigma(H_{2,t} + V_1(\cdot + te_d)), \quad t > 0. \tag{17}$$

Analogously as in the proof of Lemma 2, one can show that for every $z \in \mathbb{C}$ with $\text{dist}(z, [0, \infty))$ sufficiently large, we have strong resolvent convergence

$$(H_{2,t} + V_1(\cdot + te_d) - z)^{-1} \xrightarrow{s} (H_2 - z)^{-1}, \quad t \rightarrow \infty,$$

and the same holds for the adjoint operators; note that here we use that every $f \in C_0^\infty(\mathbb{R}^d)$ belongs to $\mathcal{D}(H_{2,t})$ for all $t > 0$ so large that $\text{supp } f \subset (\mathbb{R}_+^d - te_d)$. Therefore, by [4, Theorem 2.3 i)], in the limit $t \rightarrow \infty$ the isolated eigenvalue

$\mu \in \sigma(H_2) \setminus \sigma(H_1)$ is approximated by points $\mu_t \in \sigma(H_{2,t} + V_1(\cdot + te_d))$, $t > 0$, provided that

$$\mu \notin \sigma_{\text{ess}}((H_{2,t} + V_1(\cdot + te_d))_{t>0}) \cup \sigma_{\text{ess}}(((H_{2,t} + V_1(\cdot + te_d))^*)_{t>0})^*.$$

Similarly as in the proof of Lemma 2, one may check that the set on the right is contained in $\sigma(H_1) = \sigma(H_1^*)^*$, and $\mu \notin \sigma(H_1)$ by the assumptions. This, together with (17), proves the claim. \square

Proof of Lemma 3. First we return to the problem on the whole \mathbb{R}^d . By Lemma 1 applied to t, ε, δ and $r/2$, there exist $a > 0$ and $c \in \mathbb{C}$ such that $U_{c,t,a}$ satisfies (16), and so that the operator $-\Delta + U_{c,t,a}$ in $L^2(\mathbb{R}^d)$ has an eigenvalue μ (independent of t) with $\text{Im } \mu < 0$ and $|\mu - \lambda| < r/2$. By Lemma 4 applied to $V_1 \equiv 0$, $V_2 = U_{c,0,a}$, for every $t > 0$ sufficiently large, the operator $-\Delta + \chi_{\mathbb{R}_+^d} U_{c,t,a}$ in $L^2(\mathbb{R}_+^d)$ has an eigenvalue μ_t with $\text{Im } \mu_t < 0$ and $|\mu_t - \mu| < r/2$, hence $|\mu_t - \lambda| < r$. \square

Now the proof of the main result is straight forward.

Proof of Theorem 2. We proceed analogously as in the proof of Theorem 1 but use Lemmas 3, 4 instead of Lemmas 1, 2. Note that here t_1 is not arbitrary but given (sufficiently large) by Lemma 3. \square

Acknowledgements. The author would like to thank Ari Laptev for drawing her attention to this problem, and the referees for valuable comments. This work was supported by the Swiss National Science Foundation (SNF), Early Postdoc.Mobility project P2BEP2_159007.

REFERENCES

- [1] ABRAMOV, A. A., ASLANYAN, A., AND DAVIES, E. B. Bounds on complex eigenvalues and resonances. *J. Phys. A* **34**, 1 (2001), 57–72.
- [2] ABRAMOWITZ, M., AND STEGUN, I. A. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, vol. 55 of *National Bureau of Standards Applied Mathematics Series*. For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C., 1964.
- [3] BÖGLI, S. Convergence of sequences of linear operators and their spectra. arXiv:1604.07732, 2016.
- [4] BÖGLI, S. Local convergence of spectra and pseudospectra. to appear in *J. Spectr. Theory*. arXiv:1605.01041, 2016.
- [5] DEMUTH, M., HANSMANN, M., AND KATRIEL, G. On the discrete spectrum of non-selfadjoint operators. *J. Funct. Anal.* **257**, 9 (2009), 2742–2759.
- [6] DEMUTH, M., HANSMANN, M., AND KATRIEL, G. Lieb-Thirring type inequalities for Schrödinger operators with a complex-valued potential. *Integral Equations Operator Theory* **75**, 1 (2013), 1–5.
- [7] FRANK, R. L. Eigenvalue bounds for Schrödinger operators with complex potentials. *Bull. Lond. Math. Soc.* **43**, 4 (2011), 745–750.
- [8] FRANK, R. L., LAPTEV, A., LIEB, E. H., AND SEIRINGER, R. Lieb-Thirring inequalities for Schrödinger operators with complex-valued potentials. *Lett. Math. Phys.* **77**, 3 (2006), 309–316.
- [9] FRANK, R. L., LAPTEV, A., AND SAFRONOV, O. On the number of eigenvalues of Schrödinger operators with complex potentials. *J. Lond. Math. Soc. (2)* **94**, 2 (2016), 377–390.
- [10] FRANK, R. L., AND SIMON, B. Eigenvalue bounds for Schrödinger operators with complex potentials. II. to appear in *J. Spectr. Theory*, arXiv:1504.01144.
- [11] KATO, T. *Perturbation theory for linear operators*. Springer-Verlag, Berlin, 1995. Reprint of the 1980 edition.
- [12] LAPTEV, A. Spectral inequalities for partial differential equations and their applications. In *Fifth International Congress of Chinese Mathematicians. Part 1, 2*, vol. 2 of *AMS/IP Stud. Adv. Math.*, **51**, pt. 1. Amer. Math. Soc., Providence, RI, 2012, pp. 629–643.
- [13] LAPTEV, A., AND SAFRONOV, O. Eigenvalue estimates for Schrödinger operators with complex potentials. *Comm. Math. Phys.* **292**, 1 (2009), 29–54.

- [14] LIEB, E. H., AND THIRRING, W. E. Inequalities for the moments of the eigenvalues of the Schrödinger Hamiltonian and their relation to Sobolev inequalities. *Studies in Mathematical Physics* (1976), 269–303.
- [15] PAVLOV, B. S. On a non-selfadjoint Schrödinger operator. In *Probl. Math. Phys., No. I, Spectral Theory and Wave Processes (Russian)*. Izdat. Leningrad. Univ., Leningrad, 1966, pp. 102–132.
- [16] PAVLOV, B. S. On a non-selfadjoint Schrödinger operator. II. In *Problems of Mathematical Physics, No. 2, Spectral Theory, Diffraction Problems (Russian)*. Izdat. Leningrad. Univ., Leningrad, 1967, pp. 133–157.
- [17] PAVLOV, B. S. On a nonselfadjoint Schrödinger operator. III. In *Problems of mathematical physics, No. 3: Spectral theory (Russian)*. Izdat. Leningrad. Univ., Leningrad, 1968, pp. 59–80.
- [18] WANG, X. P. Number of eigenvalues for dissipative Schrödinger operators under perturbation. *J. Math. Pures Appl. (9)* 96, 5 (2011), 409–422.

(S. Bögli) MATHEMATISCHES INSTITUT, LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN, THERESIENSTR. 39, D-80333 MÜNCHEN
E-mail address: boegli@math.lmu.de